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Direct Gauging of the Poincaré Group. IV. Curvature, Holonomy, Spin, and Gravity

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The minimal replacement operator of direct Poincaré gauge theory converts Minkowski space-time to a new space U_4 with curvature and torsion. Explicit representations of the connection and curvature forms of U_4 are obtained. This enable us to prove that the Ricci lemma is always satisfied (U_4 is a Riemann-Cartan space) and that the holonomy group of U_4 is the component of the Lorentz group that is continuously connected to the identity. A specific form of the "free field" Lagrangian for local action of the Poincaré group is studied. Although the Lagrangian is independent of torsion, spin currents are supported by a system of algebraic relations between spin current and torsion. The field equations for the translation part of the P_{10} gauge fields are shown to be relations between the Ricci curvature and the total momentum-energy tensors, although these equations have nontrivial skew-symmetric parts whenever torsion is present. If the spin currents vanish and the total momentum-energy tensor is symmetric, Einstein's equations of general relativity with cosmological constant obtain as exact rather than approximate results. This leads to explicit evaluations of all coupling constants for the P_{10} sector, and to the fact that any solution of Einstein's field equations has the proper orthochronous Lorentz group as holonomy group. Direct gauge theory for the Poincaré group thus provides a simple and explicit method of introducing gravitational effects whenever an adequate description of matter and internal gauge structures is known on Minkowski space. An alternative system of field variables is shown to lead to a decomposition of the gravitational equations that is analogous to the decomposition of the Klein-Gordon equation via Dirac spinors.

1. INTRODUCTION

A direct gauge theory for the Poincaré group was given in paper I (Edelen, 1985a) as a specific simplification of the general structure of gauge theory based on operator-valued Lie connections (Edelen, 1984). This theory was extended in paper II (Edelen, 1985b) and the problem of gauging a Poincaré invariant theory with an internal symmetry group was examined

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in paper III (Edelen, 1985c). (Explicit citation of equations from I, II, and III will be made by hyphenation with the appropriate Roman numeral.) Although all of these results turn out to be pertinent, the theory "sits on the fence" because a specific free field Lagrangian for the total gauge group has not been specified and the curvature structure of the resulting space-time manifold has not been adequately characterized. Both of these shortcomings is rectified in this paper.

An analysis of the anholonomic structure induced on the space-time manifold U_4 by minimal replacement is given in Section 2. This leads to an explicit representation of the holonomic components of the connection coefficients of U_4, which are used in Section 3 to prove that the Ricci lemma holds for U_4. The space-time manifold is therefore a Riemann-Cartan space, and this result obtains as a direct consequence of minimal replacement for the local action of the Poincaré group. Section 4 develops a specific representation for the curvature tensor of U_4, from which it follows that the holonomy group of U_4 is the component of the Lorentz group connected to the identity element. This has far reaching implications, both theoretically and experimentally.

The question of choosing the free field Lagrangian for the Poincaré compensating fields is faced in Section 5. We restrict consideration to Lagrangians that are at most quadratic in the derivatives of field quantities. Results established in III then show that the free field Lagrangian for the total gauge group (Poincaré plus internal symmetry) is the sum of the free field Lagrangian for the Poincaré group and the lifting of the free field Lagrangian for the internal symmetry group from Minkowski space to U_4 by means of the anholonomic bases generated by minimal replacement. Thus, if the appropriate free field Lagrangian for the internal symmetry group on Minkowski space is known, it is likewise determined on U_4. In view of the lack of an experimental basis for the dependence of the Lagrangian on torsion, we take the free field Lagrangian to be independent of the torsion tensor. This does not mean that U_4 is torsion free, for the torsion tensor is still very much in evidence. Thus, since the Einstein theory of gravity has only two coupling constants, the general relativistic gravitational coupling constant and the cosmological constant, a particularly simple choice for the free field Lagrangian is made that models the Einstein-Hilbert choice.

Sections 6 and 7 derive the explicit resulting forms of the field equations for the translation and Lorentz compensating fields. The spin equations become explicit algebraic relations between the spin currents of the matter fields and the torsion fields of U_4; there are no derivatives of torsion involved and hence things are greatly simplified. This shows that a free field Lagrangian that is independent of torsion still provides for a direct geometric interpretation of intrinsic spin in terms of the torsion of U_4. There is, however, only one coupling constant involved in the spin equations, namely, the coupling constant of general relativity, and its known value is such that the predicted torsion effects become experimentally negligible for ordinary matter.

Necessary physical conditions for the reduction of U_4 to a pseudo-Riemannian space-time are shown in Section 8 to be vanishing spin currents and vanishing antisymmetric part of the total momentum-energy tensor. These are exactly the conditions under which the Einstein field equations of general relativity were derived, and the field equations for the direct gauge theory of the Poincaré group reduce exactly to the Einstein field equations in this case. This exact reduction also serves to determine the two coupling constants of the theory in terms of the general relativistic gravitational constant and the cosmological constant. There is an added payoff here, however, for we know that U_4 has the Lorentz group as its holonomy group and hence any solution of the Einstein field equations has the Lorentz group as its holonomy group. Thus, parallel translation of any vector around a closed circuit is equivalent to a proper, orthochronous Lorentz transformation of the vector.

2. ANHOLONOMIC STRUCTURES INDUCED BY MINIMAL REPLACEMENT

Minimal replacement for the Poincaré group [see (I-13) and (I-60)] applied to the natural bases $\{dx^i, \partial_i\}$ on M_4 gives

$$\mathcal{M}(dx^i) = B^i = B^i_j\, dx^j = (\delta^i_j + W^\alpha_j 1_\alpha{}^i{}_k x^k + \phi^i_j)\, dx^j \tag{1}$$

$$\mathcal{M}(\partial_i) = b_i = b^j_i\, \partial_j \tag{2}$$

where

$$b^i_j B^j_k = B^i_j b^j_k = \delta^i_k \tag{3}$$

Now, minimal replacement for P_{10} carries quantities from Minkowski space M_4 to a new space U_4 with the same coordinate cover and hence the natural bases $\{dx^i, \partial_i\}$. Accordingly, we may view the quantities $\{B^i, b_i\}$ as gauge anholonomic bases for U_4. Interpreted in this way, (1)-(3) give the following relations between the natural and anholonomic bases:

$$dx^i = b^i_j B^j, \qquad B^i = B^i_j\, dx^j \tag{4}$$

$$\partial_i = B^j_i b_j, \qquad b_i = b^j_i\, \partial_j \tag{5}$$

Quantities in U_4 referred to the anholonomic bases $\{B^i, b_i\}$ will be designated by a superimposed "hat" (^) and referred to as gauge anholonomic components. We thus have

$$\alpha = \alpha_i \, dx^i = \hat{\alpha}_i B^i, \qquad v = v^i \partial_i = \hat{v}^i b_i \tag{6}$$

with

$$\hat{\alpha}_i = \alpha_j b^j_i, \qquad \alpha_i = B^j_i \hat{\alpha}_j, \qquad \hat{v}^i = B^i_j v^j$$
$$v^i = \hat{v}^j b^i_j \tag{7}$$

The simplest example of this occurs with regard to the metric structure of U_4. The results given in (I-14) show that

$$dS^2 = \mathcal{M}(h_{ij} \, dx^i \otimes dx^j) = h_{ij} B^i \otimes B^j = g_{ij} \, dx^i \otimes dx^j$$

where

$$g_{ij} = B^k_i h_{km} B^m_j, \qquad g^{ij} = b^i_k h^{km} b^j_k \tag{8}$$

We thus have

$$\hat{g}_{ij} = h_{ij}, \qquad \hat{g}^{ij} = h^{ij} \tag{9}$$

that is, the gauge anholonomic components of the metric on U_4 are the components of the metric on M_4.

Next, we note that the differential system generated by the 1-forms B^i [see (I-26)] is $DB_i = \Sigma^i$. When these equations are written out, we have

$$dB^i + W^\alpha_k l_{\alpha j}^{\ i} \, dx^k \wedge B^j = \Sigma^i$$

If (4) is used, these equations become

$$dB^i + W^\alpha_r b^r_k l_{\alpha j}^{\ i} B^k \wedge B^j = \Sigma^i \tag{10}$$

Equations (10) are, however, nothing more than the equations of structure of E. Cartan associated with the basis $\{B^i | 1 \le i \le 4\}$ for $\Lambda^1(U_4)$. In view of the skew symmetry of the second set of terms on the left-hand side of (10) in the indices k, j, these equations serve to identify the anholonomic components of the connection coefficients on U_4 by

$$\hat{\Gamma}^{\ i}_{kj} = \hat{W}^\alpha_k l_{\alpha j}^{\ i} + \hat{Y}^i_{kj}, \qquad \hat{Y}^i_{[kj]} = 0 \tag{11}$$

On the other hand, the connection 1-forms induced by minimal replacement have the evaluation

$$W^\alpha_k l_{\alpha j}^{\ i} \, dx^k = \hat{W}^\alpha_m l_{\alpha j}^{\ i} B^m$$

from which (10) were obtained. Comparison with (11) shows that all of the Y's must be zero and that minimal replacement induces connection coefficients on U_4 referred to the anholonomic basis generated by the B's and the b's. Thus, minimal replacement lifts all geometric quantities from

M_4 up to U_4, but the results are referred to the anholonomic bases generated by the B's and the b's.

The relations between the connection components and their anholonomic resolutions are well known (Schouten, 1954):

$$\Gamma^i_{kj} = B^q_k B^r_j b^i_p \hat{\Gamma}^p_{qr} + b^i_p \,\partial_k B^p_j$$

Accordingly, when (11) is used with all of the Y's set to zero, we obtain

$$\Gamma^i_{kj} = W^\alpha_k L_{\alpha}{}^i_j + b^i_p \,\partial_k B^p_j \tag{12}$$

where we have set

$$L^i_{\alpha j} = b^i_p l_{\alpha}{}^p_q B^q_j \tag{13}$$

Thus, the operation of minimal replacement for P_{10} induces a connection on U_4 that is uniquely determined by the compensating 1-forms W^α for the Lorentz sector and the distortion 1-forms B^i.

Contrary to previous practices, (12) shows that the components of connection on U_4 do not take their values in the Lie algebra of P_{10}. There is a partial correspondence, however, which we proceed to derive. First, it follows directly from (13) that

$$L_{\alpha}{}^i_j L_{\beta}{}^j_k - L_{\beta}{}^i_j L_{\alpha}{}^j_k = C_{\alpha}{}^\gamma_\beta L_{\gamma}{}^i_k \tag{14}$$

while (8), (11), (12), and (13) give

$$L_{\alpha}{}^j_{(i} g_{k)j} = 0, \qquad L^{(i}_{\alpha j} g^{k)j} = 0 \tag{15}$$

The six matrices $\{\mathbf{L}_\alpha | 1 \leq \alpha \leq 6\}$ thus form a matrix representation of the Lie algebra of the Lorentz group on U_4. The first set of terms in the representation (12) of the connection coefficients on U_4 thus take their values in the Lie algebra of P_{10}; it is just that the terms involving derivatives of the B^i's do not.

This result is not a singular happenstance, for other algebraic structures have similar liftings to U_4. For instance, if $(\boldsymbol{\gamma}^i | 1 \leq i \leq 4\}$ is a basis for the Dirac algebra on M_4 and $\boldsymbol{\Psi}$ is the corresponding spinor wave function, the results of I and III show that

$$\mathcal{M}(\boldsymbol{\gamma}^i \,\partial_i \boldsymbol{\Psi}) = \boldsymbol{\gamma}^i b^k_i D_k \boldsymbol{\Psi} = \boldsymbol{\sigma}^k D_k \boldsymbol{\Psi} \tag{16}$$

where we have set

$$\boldsymbol{\sigma}^k = b^k_i \boldsymbol{\gamma}^i \tag{17}$$

It is then an easy matter to verify that

$$\boldsymbol{\sigma}^i \boldsymbol{\sigma}^j + \boldsymbol{\sigma}^j \boldsymbol{\sigma}^i = 2 g^{ij} \mathbf{I} \tag{18}$$

and hence the $\boldsymbol{\sigma}$'s form a basis for the Dirac algebra on U_4.

3. METRICITY OF THE GAUGE-INDUCED CONNECTION

We now know that minimal replacement for P_{10} lifts Minkowski space to a new space U_4 whose metric and connection are uniquely determined in terms of the compensating fields for the local action of P_{10}. This new space is subject to arbitrary smooth changes of coordinate covers, because of the local action of the translation subgroup, and admits matrix representations of the Dirac algebra and the Lie algebra of $L(4, R)$. Since both the metric and the connection are determined once the compensating fields are known, the question arises as to whether the Ricci lemma holds for U_4.

Let ∇_i denote the operation of covariant differentiation based on the connection of U_4 given by (12). Following Schouten (1954), the metricity tensor of U_4 is defined by

$$Q_{kij} = -\nabla_k g_{ij} \tag{19}$$

If this tensor vanishes on U_4, the connection (12) is metric, in which case raising and lowering of indices commute with covariant differentiation and U_4 is a Riemann–Cartan space. The problem is therefore that of substituting (8) and (12) into the right-hand side of (19) in order to compute the metricity, a lengthy and involved task.

An alternative approach that is significantly simpler obtains from the observation that

$$Q_{kij} = B_k^a B_i^p B_j^q \hat{Q}_{apq} \tag{20}$$

where

$$\hat{Q}_{apq} = b_a^k b_p^i b_q^j Q_{kij} \tag{21}$$

We now substitute (19) into (21) and obtain

$$\hat{Q}_{apq} = -\hat{\nabla}_a \hat{g}_{pq} \tag{22}$$

where

$$\hat{\nabla}_a = b_a^k \nabla_k$$

is the symbol for covariant differentiation based upon the anholonomic components of the connection. Accordingly, (9), (11), and (22) give

$$\hat{Q}_{apq} = -b_a^k(\partial_k h_{pq} - 2 W_k^\alpha l_{\alpha(p}{}^i h_{q)i}) = 0 \tag{23}$$

The metricity tensor of U_4 thus vanishes.

The connection induced on U_4 by minimal replacement is metric,

$$\nabla_k g_{ij} = 0, \qquad \hat{\nabla}_a \hat{g}_{pq} = \hat{\nabla}_a h_{pq} = 0 \tag{24}$$

and U_4 is a Riemann–Cartan space.

Previous works have used the Ricci conditions, $Q_{ijk}=0$, as imposed conditions in order to obtain representations for the connection coefficients because they use a different metric tensor (see I, Section 1) and different anholonomic bases. Underlying these differences is the fact that previous works have not used the minimal replacement operator to induce the appropriate structure on U_4; rather, the geometry is put in first by requiring U_4 to be a Riemann–Cartan space,[2] and only afterwards are arguments by analogy used to introduce gaugelike constructs. The approach taken here is to start with the physics, as described by the Lagrangian for the matter fields on Minkowski space, and then simply allow the appropriate structures to unfold through the action of the minimal replacement operator induced by the local action of the Poincaré group. We thus have significantly fewer mathematical assumptions and yet we obtain all of the necessary structure on U_4 including the Ricci lemma without having to explicitly put it in as an independent system of conditions.

4. CURVATURE CONSIDERATIONS

We shall need certain relations between the holonomic and anholonomic components of the curvature tensor of U_4. Following Schouten (1954), the curvature 2-forms of U_4 are given by

$$R^i_j=\tfrac{1}{2}R^i_{kmj}\,dx^k\wedge dx^m=d\Gamma^i_j+\Gamma^i_r\wedge\Gamma^r_j \tag{25}$$

with

$$\Gamma^i_j=\Gamma^i_{qj}\,dx^q$$

Let us set

$$\tilde{R}^a_{kmp}=R^i_{kmj}B^a_i b^j_p \tag{26}$$

We then have

$$\tilde{R}^a_p=\tfrac{1}{2}\tilde{R}^a_{kmp}\,dx^k\wedge dx^m,\qquad \tilde{\Gamma}^a_p=\hat{\Gamma}^a_{rp}B^r_s\,dx^s \tag{27}$$

and the Cartan equations of structure give

$$\tilde{R}^a_p=d\tilde{\Gamma}^a_p+\tilde{\Gamma}^a_r\wedge\tilde{\Gamma}^r_p$$

Now, (11) and (27) show that

$$\tilde{\Gamma}^a_p=W^\alpha l_{\alpha\,p}^{\;a} \tag{28}$$

and hence (I-21) gives

$$\tilde{R}^a_p=\theta^\alpha l_{\alpha\,p}^{\;a}$$

[2]If the Y's are retained in (11), the resulting Q's vanish if and only if the Y's vanish.

We now substitute this result into (26) and obtain

$$R^{i}_{mnj} = \theta^{\alpha}_{mn} l_{\alpha}{}^{a}{}_{p} B^{p}_{j} b^{i}_{a} = \theta^{\alpha}_{mn} L_{\alpha}{}^{i}_{j}$$

where the latter equality follows from (13).

The representation of the curvature tensor of U_4 in the form

$$R^{i}_{mnj} = \theta^{\alpha}_{mn} L_{\alpha}{}^{i}_{j} \tag{29}$$

is of particular significance. We have already seen that $\{\mathbf{L}_\alpha | 1 \leq \alpha \leq 6\}$ forms a matrix representation of the Lie algebra of the Lorentz group on U_4 [see (14)]. The space U_4 has a much richer collection of admissible transformations than just those generated by the local action of $T(4)$ (coordinate transformations), however. We also have the frame and coframe transformations generated by the local action of $L(4, R)$. Thus, $\boldsymbol{\theta} = \theta^\alpha \mathbf{l}_\alpha$ and (I-31) give

$$'\boldsymbol{\theta} = {}'\theta^\alpha \mathbf{l}_\alpha = \mathbf{L}\boldsymbol{\theta}\mathbf{L}^{-1}$$

and hence

$$'\theta^{\alpha}_{mn} = G^{\alpha}_{\beta} \theta^{\beta}_{ij} \frac{\partial x^i}{\partial' x^m} \frac{\partial x^j}{\partial' x^n} \tag{30}$$

Here, the G's constitute the matrix representation of the adjoint action of $L(4, R)$ on its Lie algebra that is defined by

$$\mathbf{L}\mathbf{l}_\alpha \mathbf{L}^{-1} = G^{\beta}_{\alpha} \mathbf{l}_\beta$$

In like manner, $'\mathbf{B} = \mathbf{L}\mathbf{B}$, $'\mathbf{b} = \mathbf{b}\mathbf{L}^{-1}$ and (13) show that

$$'L_{\alpha}{}^{i}_{j} = \overset{-1}{G}{}^{\beta}_{\alpha} L_{\beta}{}^{p}_{q} \frac{\partial' x^i}{\partial x^p} \frac{\partial x^q}{\partial' x^j} \tag{31}$$

Accordingly, (29), (30), and (31) show that the curvature tensor on U_4 only responds to the coordinate changes achieved by local action of the translation group, as indeed it must.

Now that we know how the $\mathbf{L}_\alpha$'s transform under the local action of P_{10}, we may proceed to compute its total covariant derivative:

$$\overset{T}{\nabla}_k L_{\alpha}{}^{i}_{j} = \partial_k L_{\alpha}{}^{i}_{j} - \Gamma_{k\alpha}{}^{\beta} L_{\beta}{}^{i}_{j} - \Gamma_{kj}{}^{m} L_{\alpha}{}^{i}_{m} + \Gamma_{k}{}^{i}_{m} L_{\alpha}{}^{m}_{j}$$

Since the connection coefficients for the adjoint action of $L(4, R)$ are given by (Rund, 1982; Edelen, 1984)

$$\Gamma_{k\alpha}^{\ \ \beta} = W^{\gamma}_{k} C_{\gamma\alpha}^{\ \ \beta}$$

a direct calculation using (12) and (14) shows that

$$\overset{T}{\nabla}_k L_{\alpha}{}^{i}_{j} = 0 \tag{32}$$

Thus, the $\mathbf{L}_\alpha$'s form a covariant constant representation of the Lie algebra of $L(4, R)$ on U_4.

It is clear from the definition of the total covariant derivative that it agrees with the ordinary covariant derivative when acting on quantities that only respond to the general coordinate transformations generated by local action of $T(4)$. Thus, since (II-32) shows that the metric tensor only responds to coordinate transformations, we have

$$\overset{T}{\nabla}_k g_{ij} = \nabla_k q_{ij} = 0$$

Further,

$$`B^i_j = L^i_k B^k_m \frac{\partial x^m}{\partial `x^j}, \qquad `b^i_j = \frac{\partial `x^i}{\partial x^k} b^k_m \overset{-1}{L}{}^m_j$$

and use of (12) and (28) show that

$$\overset{T}{\nabla}_k B^i_j = \partial_k B^i_j - \Gamma^m_{kj} B^i_m + \tilde{\Gamma}^i_{km} B^m_j = 0$$

$$\overset{T}{\nabla}_k b^i_j = \partial_k b^i_j + \Gamma^i_{km} b^m_j - \tilde{\Gamma}^m_{kj} b^i_m = 0$$

On the other hand, it is easily seen that

$$\nabla_k B^i_j \neq 0, \qquad \nabla_k b^i_j \neq 0$$

because the B^i's are 1-forms and the b_j's are vectors as far as coordinate transformations of U_4 are concerned. Finally, we note that

$$\nabla_k R_{mnj}^{\;\;\;\;\;\,i} = \overset{T}{\nabla}_k R_{mnj}^{\;\;\;\;\;\,i} = (\overset{T}{\nabla}_k \theta^\alpha_{mn}) L_{\alpha j}^{\;\;\;\,i}$$

when (29) and (32) are used.

The known properties of the holonomy group of a manifold with a linear connection (Schouten, 1954, pp. 375ff; Hlavatý, 1959) show that the factorization given by (29) implies that the $\mathbf{L}_\alpha$'s form a basis for the Lie algebra of the holonomy group of U_4. Accordingly, the holonomy group for the space U_4, which obtains from minimal replacement for the Poincaré group, is the component of the Lorentz group that is continuously connected to the identity. The Lorentz group is thus always present as the holonomy group in the U_4 of direct Poincaré gauge theory, and this is true irrespective of whether U_4 does or does not have vanishing torsion. In particular, the Lorentz group is the holonomy group for any gravitational theory that obtains via the Poincaré gauge theory.

The definition of the Ricci tensor and (29) give

$$R_{ij} = R_{mij}^{\;\;\;\;\;\,m} = \theta^\alpha_{mj} L_{\alpha j}^{\;\;\;\,m} \tag{33}$$

and hence the scalar curvature $R = g^{ij}R_{ij}$ has the evaluation

$$R = \theta^{\alpha}_{mi} g^{ij} L_{\alpha j}^{\ m} \tag{34}$$

When (3), (8), and (13) are used, a straightforward calculation shows that

$$R = b_a^{[m} b_r^{n]} \theta^{\alpha}_{mn} l_{\alpha s}^{\ a} h^{rs} = \alpha_1 \tag{35}$$

where α_1 is the P_{10} invariant given in II.

The other independent contraction of the full curvature tensor has the evaluation

$$V_{mn} = R^{i}_{mni} = \theta^{\alpha}_{mn} L_{\alpha i}^{\ i} = 0 \tag{36}$$

because (13) gives

$$L_{\alpha i}^{\ i} = b_a^i l_{\alpha p}^{\ a} B_i^p = l_{\alpha a}^{\ a} = 0 \tag{37}$$

5. CHOICE OF THE FREE FIELD LAGRANGIAN

The theory starts with the Lagrangian for the matter fields to which the operation of minimal replacement is applied in order to obtain the Lagrangian BL which is invariant under the local action of the Poincaré group [remember that $B = \det(\mathbf{B}) = (-g)^{1/2}$]. This is only part of the story, for we still have to choose the invariant "free field" Lagrangian V for the local action of the total group $P_{10} \times G$, where G is the internal symmetry group of the matter fields. If we follow the customary procedure of restricting the Lagrangian to be at most quadratic in the derivatives of the field variables, the results established in II and III show that

$$V = B(\Pi_P + \Pi_G) \tag{38}$$

Here,

$$\Pi_P = k_0 + k_1\alpha_1 + k_2(\alpha_1)^2 + k_3\alpha_2 + k_4\beta_1 + k_5\rho_1 + k_6\eta_1 \tag{39}$$

is a P_{10}-invariant scalar-valued function [see (II-85)], while (III-20) gives

$$\Pi_G = U(g_{ij}, N^a_{ij}) = k_7 N^b_{ij} g^{ik} g^{jm} N^c_{km} k_{bc} \tag{40}$$

Here, we have used N's instead of θ's to denote the components of the curvature 2-forms formed from the compensating 1-forms for the local action of the internal symmetry group G in order that they not be confused with curvature components associated with P_{10}. The problem thus boils down to choosing the eight coupling constants $k_0, k_1, \ldots, k_7$.

The required choices cannot be made in a conceptual vacuum, and hence it is useful at this point to write out the pertinent field equations developed in I and II. Since our primary concern is with deciding between the various terms in (39), we shall ignore for the moment the field equations given in III for the matter fields and the compensating fields of the internal symmetry group. We thus have the constitutive relations

$$L_A^i = \partial L/\partial y_i^A, \qquad y_i^A = \mathcal{M}(\partial_i \Psi^A) \tag{41}$$

$$G_k^{ij} = \partial V/\partial \Sigma_{ij}^k, \qquad H_\alpha^{ij} = (\partial V/\partial \theta_{ij}^\alpha)|_\Sigma \tag{42}$$

$$T_j^i = L_A^i y_j^A - \delta_j^i L \tag{43}$$

$$S_k^i = (\partial V/\partial \phi_i^k)|_{\theta,\Sigma} = (\partial V/\partial B_i^k)|_{\theta,\Sigma} \tag{44}$$

where the last equality follows from

$$B_i^k = \delta_i^k + W_i^\alpha l_{\alpha j}^{\ k} x^j + \phi_i^k$$

The field equations for the P_{10} gauge fields then take the form

$$d\{G_k^{ij}\mu_{ij}\} - W^\alpha l_{\alpha k}^{\ m} \wedge \{G_m^{ij}\mu_{ij}\} = (S_k^i - Bb_j^i T_k^j)\mu_i, \qquad 1 \le k \le 4 \tag{45}$$

$$\begin{aligned} d\{H_\alpha^{ij}\mu_{ij}\} - W^\gamma C_{\gamma\ \alpha}^{\ \beta} \wedge \{H_\beta^{ij}\mu_{ij}\} &= Bb_k^i L_A^k M_{\alpha\ B}^{\ A} \Psi^B \mu_i \\ &\quad - B^k l_{\alpha k}^{\ m} \wedge \{G_m^{ij}\mu_{ij}\} \qquad 1 \le \alpha \le 6 \end{aligned} \tag{46}$$

The first thing we note is that the Poincaré gauge fields contribute an orbital spin current

$$-B^k l_{\alpha k}^{\ m} \wedge \{G_m^{ij}\mu_{ij}\}$$

unless the G's vanish. This can be the case, in view of (42), only if Π_P does not depend on the components of the Cartan torsion. In like manner, there seems to be no definitive experimental evidence that physical space-time has nonvanishing Cartan torsion, and hence a torsion contribution to the free field Lagrangian would appear to be beyond experimental verification. Accordingly, since only ρ_1 and η_1 depend on the torsion (see II, Section 5), we take

$$k_5 = k_6 = 0 \tag{47}$$

Careful note should be taken that (47) does not assume that the Cartan torsion vanishes; we have only assumed that the free field Lagrangian does not depend on the Cartan torsion.

We have already seen that $\alpha_1 = R$, where R is the curvature scalar of U_4. If the Cartan torsion vanishes, then R is the scalar curvature of a pseudo-Riemannian manifold, in which case $BR = (-g)^{1/2}R$ is known to be the Lagrangian function for gravitation in general relativity. Although renormalization considerations might suggest that there should be terms in Π_P that are quadratic in the curvature components, these terms would have to be quite small relative to the R term in order not to lead to predictions in conflict with the experimental basis for general relativity. Accordingly, it seems useful to examine the case where the free field Lagrangian is given by

$$V = B(k_0 + k_1\alpha_1 + U) \tag{48}$$

6. THE TRANSLATION EQUATIONS

We first take up the field equations (45) that obtain from variation of the compensating fields for the translation sector. With V given by (48), (42) shows that

$$G_k^{ij} = 0$$

The field equations (45) thus reduce to

$$S_k^i = Bb_j^i T_k^j \tag{49}$$

The problem thus boils down to evaluating the S's.

Because V has the form $B\Pi$, (44) gives

$$S_k^i = B(\Pi b_k^i + \partial\Pi/\partial B_i^k) \tag{50}$$

Now, Π depends on the B's only through the b's, and hence

$$\partial b_r^m/\partial B_i^k = -b_k^m b_r^i$$

gives

$$S_k^i = Bb_m^i(\Pi\delta_k^m - b_k^r\,\partial\Pi/\partial b_m^r) \tag{51}$$

We now note that $\Pi = \Pi_P + U$, where

$$\Pi_P = k_0 + k_1 b_a^t b_r^n \theta_{tn}^\alpha l_{\alpha s}{}^a h^{rs} \tag{52}$$

and that U depends on the b's only through the quantities g^{ij} [see (40)]. A direct calculation using the skew symmetry of

$$\theta_{ij}^\alpha \qquad \text{and} \qquad l_{\alpha\ m}^{\ i} h^{mj}$$

in the indices i, j then yields

$$S_k^i = Bb_j^i\left\{k_0\delta_k^i - 2k_1(\hat{R}_{ks} - \tfrac{1}{2}\hat{R}h_{ks})h^{sj} + \frac{1}{B}\frac{\partial(BU)}{\partial g_{rs}} B_r^j B_s^b h_{bk}\right\} \tag{53}$$

We now substitute (53) into (49) and obtain

$$T_k^j = k_0\delta_k^j - 2k_1(\hat{R}_{ks} - \tfrac{1}{2}\hat{R}h_{ks})h^{sj} + \frac{1}{B}\frac{\partial(BU)}{\partial g_{rs}} B_r^j B_s^b h_{bk} \tag{54}$$

The first thing to be noted is that these equations refer to tensorial quantities resolved on the anholonomic bases generated by the B's and the b's. As such, they are scalar equations on U_4, and hence are the easiest form of the field equations to use when it comes down to actually obtaining solutions. On the other hand, identification is only directly available when the field equations are written in terms of tensors referred to holonomic coordinate covers. Thus, if we lower the j index by multiplying by h_{ji}, and then use (7), the field equations become

$$-2k_1(R_{ab}-\tfrac{1}{2}Rg_{ab})+k_0g_{ab}=T^i_k h_{ji}B^k_a B^i_b-\frac{1}{B}\frac{\partial(BU)}{\partial g_{rs}}g_{ra}g_{sb}=\tau_{ab} \quad (55)$$

The tensor τ_{ab} is clearly the the total momentum-enery tensor since it is the sum of the momentum-energy tensor for the matter fields and that for the internal symmetry group G. Thus, if we set

$$\kappa=-(2k_1)^{-1}, \qquad \lambda=\kappa k_0 \quad (56)$$

then (55) assumes the form

$$R_{ij}-\tfrac{1}{2}Rg_{ij}+\lambda g_{ij}=\kappa\tau_{ij} \quad (57)$$

These equations are a bit misleading, however, for we are in a Riemann-Cartan space U_4 with torsion, and hence R_{ij} is not symmetric. Thus, if we take the symmetric and skew-symmetric parts of (57), we finally obtain

$$R_{(ij)}-\tfrac{1}{2}Rg_{ij}+\lambda g_{ij}=\kappa\tau_{(ij)} \quad (58)$$

and

$$R_{[ij]}=\kappa\tau_{[ij]} \quad (59)$$

where $R_{[ij]}$ is uniquely determined in terms of the holonomic components of the torsion tensor of U_4 and its covariant derivative in view of (36) (see Schouten, 1954, p. 144).

Antisymmetric momentum-energy tensors do arise in practice. For a term in the matter Lagrangian of the form

$$\bar{\Psi}\gamma^i\partial_i\Psi$$

the covariant form of the momentum-energy tensor before minimal replacement is given by

$$\bar{\Psi}\gamma^k(h_{ki}\,\partial_j\Psi-h_{ij}\,\partial_k\Psi)$$

which is not symmetric in i, j.

7. THE SPIN EQUATIONS

We now turn to the field equations (46), which reduce to

$$d\{H^{ij}_{\alpha}\mu_{ij}\} - W^{\gamma}C_{\gamma}{}^{\beta}{}_{\alpha} \wedge \{H^{ij}_{\beta}\mu_{ij}\} = Bb^{i}_{k}L^{k}_{A}M_{\alpha}{}^{A}_{B}\Psi^{B}\mu_{i} = J^{i}_{\alpha}\mu_{i} \tag{60}$$

because the G's all vanish. Here the J's denote the total spin currents of the matter fields. If the indicated exterior derivative and product on the left-hand side of (60) are evaluated explicitly, the field equations become

$$2(\partial_{i}H^{ij}_{\alpha} - W^{\gamma}_{i}C_{\gamma}{}^{\beta}{}_{\alpha}H^{ij}_{\beta}) = J^{j}_{\alpha} \tag{61}$$

When (48) is substituted into (42), the H's are given by

$$H^{ij}_{\alpha} = k_{1}Bl_{\alpha}{}^{p}_{s}b^{i}_{p}b^{j}_{q}h^{sq} = k_{1}Bg^{kj}L_{\alpha}{}^{i}_{k} \tag{62}$$

The H's are thus the components of a skew-symmetric contravariant tensor densities on U_4. Further, we see directly from (62) that

$$\overset{T}{\nabla}_{k}H^{ij}_{\alpha} = 0, \qquad \overset{T}{\nabla}_{i}H^{ij}_{\alpha} = 0 \tag{63}$$

because each term on the right-hand side of (62) is covariant constant on U_4. Now, direct expansion of the indicated covariant derivatives gives

$$\overset{T}{\nabla}_{i}H^{ij}_{\alpha} = \partial_{i}H^{ij}_{\alpha} - \Gamma^{\beta}_{i\alpha}H^{ij}_{\beta} + 2\Gamma^{i}_{[im]}H^{mj}_{\alpha} + \Gamma^{j}_{[im]}H^{im}_{\alpha} = 0 \tag{64}$$

and hence we have

$$\partial_{i}H^{ij}_{\alpha} - W^{\gamma}_{i}C_{\gamma\alpha}^{\beta}H^{ij}_{\beta} = -\Gamma^{i}_{[im]}H^{im}_{\alpha} - 2\Gamma^{i}_{[im]}H^{mj}_{\alpha} \tag{65}$$

when (15) is used. Accordingly, when the field equations (62) are used, the spin equations become equivalent to

$$\Gamma^{j}_{[im]}H^{im}_{\alpha} + 2\Gamma^{i}_{[im]}H^{mj}_{\alpha} = -\tfrac{1}{2}J^{j}_{\alpha} \tag{66}$$

Noting that

$$S^{i}_{jk} = \Gamma^{i}_{[jk]} = \Sigma^{r}_{jk}b^{i}_{r} \tag{67}$$

is the torsion tensor of U_4 in holonomic frames, (62), (56), and (66) combine to give

$$g^{km}(S^{j}_{im}L_{\alpha}{}^{i}_{k} - 2S^{i}_{im} = \frac{\kappa}{B}J^{j}_{\alpha} \tag{68}$$

These are the final form for the spin equations in a U_4 with Lagrangian (48). It follows directly from these equations that any one nonvanishing component of the spin current implies that U_4 has nonvanishing torsion. Thus, in particular, U_4 can reduce to a pseudo-Riemannian space only if all spin currents vanish throughout U_4. In marked contrast with previously published gauge theories for the Poincaré group, the spin equations are seen to be a system of 24 algebraic rather than differential equations for the determination of the 24 independent components of the torsion tensor [see the Appendix for explicit solution of (68)]. Similar direct relations between torsion and spin quantities have surfaced in the past (Finkelstein, 1960; Israel and Trollope, 1961; Finkelstein and Ramsy, 1962) in different contests, for they provide useful and direct interpretations of the intrinsic change in the space-time manifold that results from matter with spin degrees of freedom.

The important thing to note here is that spin is adequately accounted for in terms of torsion even though the free field Lagrangian V does not depend on the torsion. We shall see in the next section that the coupling constant κ has the value of the gravitational coupling constant of general relativity and is thus quite small. The components of the torsion in the presence of nonzero spin currents are thus correspondingly small by (68), which probably accounts for the lack of experimental detection of torsion effects in physical space-time.

8. GRAVITATION AND EVALUATION OF THE COUPLING CONSTANTS

A Riemann–Cartan space reduces to a pseudo-Riemannian space only when the torsion tensor vanishes through the space. We have just seen, however, that the presence of any one nonzero component of the spin currents requires nonvanishing torsion, and hence the torsion can vanish only if the matter fields have spin currents that vanish everywhere.

We also know that the Ricci tensor R_{ij} of a pseudo-Riemannian space is symmetric. Thus, (60) shows that U_4 can reduce to a pseudo-Riemannian space only if the total momentum-energy tensor is symmetric. Thus, necessary physical conditions for the reduction of U_4 to a pseudo-Riemmanian space are

$$\tau_{[ij]}=0, \qquad J^i_{\ \alpha}=0 \tag{69}$$

The corresponding mathematical conditions are vanishing of all components of the Cartan torsion tensor, which we have seen can be achieved through an appropriate choice of the translation compensating fields (see II, Section 7).

If these conditions are satisfied, the only surviving field equations are

$$R_{ij}-\tfrac{1}{2}Rg_{ij}+\lambda g_{ij}=\kappa\tau_{ij} \tag{70}$$

which are just Einstein's field equations with cosmological constant. The Einstein theory thus obtains exactly, and exactly in those circumstances in which it was originally derived; namely, for matter fields with symmetric momentum-energy tensors and vanishing spin currents.

We have only the one free field Lagrangian V given by (48), and hence it will be the same regardless of the form of the Lagrangian for the matter fields. If the matter field Lagrangian is such that the conditions (69) are satisfied, the only surviving field equations of the theory are Einstein's field equations for the gravitational field. Now, the constants κ and λ in the Einstein field equations have specific values: λ is the cosmological constant whose value is potentially measurable from observations of very-large-scale distributions of matter, while κ has the evaluation $8\pi\gamma/c^4$ with γ the ordinary constant of gravitation. When these evaluations are substituted into (56), we obtain

$$k_1 = \frac{-c^4}{16\pi\gamma}, \qquad k_0 = \frac{\lambda c^4}{8\pi\gamma} \tag{71}$$

The coupling constants for the theory are thus fixed once and for all provided the free field Lagrangian BU for the internal symmetry group of the matter fields is already known from the formulation of gauge theory for the internal symmetry group G on Minkowski space.

When the physical conditions (69) are satisfied, the direct gauge theory for the Poincaré group with Lagrangian (48) leads to a pseudo-Riemannian space whose geometric structure is determined by solving the Einstein field equations of general relativity. We have seen, however, that any space-time obtained by the direct Poincaré gauge theory admits $L(4, R)$ as its holonomy group. We therefore have the following important result. Any solution of the Einstein field equations gives a space-time whose holonomy group is the component of the Lorentz group that is continuously connected to the identity.

The fundamental question of how to turn on gravity in a theory that is both mathematically and physically well posed on Minkowski space would appear to be answered by the results obtained here. In this regard, we take particular note that there are no adjustable constants floating around. The theory is thus closed whenever the total Lagrangian for the matter fields and compensating fields for the internal symmetry group is known on M_4. Further, and of possibly greater importance, torsion and curvature effects in the theory are fully accounted for in terms of the field variables

$$W_i^\alpha \qquad \text{and} \qquad B_j^i \tag{72}$$

since the ϕ's can be eliminated in favor of the B's through (1). Viewed from this perspective, curvature and torsion are first-order rather than second-order differential concomitants of the field variables and significant simplifications result. This situation is somewhat analogous to replacing the second-order Klein–Gordon equation by the Dirac equations, for we have the transition from second-order equations in q_{ij} in Einstein gravitational theory to first-order equations in the list (72). This formulation gives the system of field equations (70), where the Ricci tensor is now written in terms of the W's and B's,

$$\Sigma^{i}_{jk} = 0 \tag{73}$$

and six gauge conditions for the six W's, for a total of 40 independent equations for the 40 field variables (72) for spin-free matter with a symmetric momentum–energy tensor (i.e., in the classical gravitational setting).

APPENDIX

The purpose of this appendix is to obtain the explicit representation for the torsion tensor of U_4 that is implied by the spin equations, (68). The first step is to define the auxiliary quantities

$$L_{\alpha}{}^{ij} = L_{\alpha}{}^{i}{}_{k} g^{kj}, \qquad L_{\alpha ij} = g_{ik} L_{\alpha}{}^{k}{}_{j} \tag{A1}$$

so that

$$L_{\alpha}{}^{(ij)} = 0, \qquad L_{\alpha(ij)} = 0 \tag{A2}$$

A straightforward calculation then shows that

$$2L_{\alpha}{}^{ij} C^{\alpha\beta} L_{\beta km} = \delta^{i}_{m}\, \delta^{j}_{k} - \delta^{i}_{k}\, \delta^{j}_{m} \tag{A3}$$

We now use (A1) to write (68) in the equivalent form

$$BS^{j}_{im} L_{\alpha}{}^{im} = \kappa J^{j}_{\alpha} + 2BS^{i}_{im} L_{\alpha}{}^{jm} \tag{A4}$$

When these relations are multiplied by $C^{\alpha\beta} L_{\beta kr}$ and summed on α, use of (A3) leads to the evaluations

$$BS^{j}_{kr} = -\kappa J^{j}_{\alpha} C^{\alpha\beta} L_{\beta kr} - B(S^{i}_{ik}\, \delta^{j}_{r} - S^{i}_{ir}\, \delta^{j}_{k}) \tag{A5}$$

Thus, a contraction on j and k gives

$$2BS^{i}_{ir} = \kappa J^{i}_{\alpha} C^{\alpha\beta} L_{\beta ir} \tag{A6}$$

and hence (A5) gives the desired solution

$$BS^{j}_{kr} = \kappa C^{\alpha\beta} (L_{\beta kr}\, \delta^{j}_{i} + L_{\beta i[k}\, \delta^{j}_{r]}) J^{i}_{\beta} \tag{A7}$$

The explicit solution given by (A7) shows that the torsion of U_4 is nonzero only at those space-time points where the spin currents are nonzero. Thus, torsion does not propagate in space-time, and any region of U_4 for which the spin currents vanish is a pseudo-Riemannian region.

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